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Relations between Perron–Frobenius results for matrix pencils

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Abstract

Two different generalizations of the Perron–Frobenius theory to the matrix pencil $Ax = \lambda Bx$ are discussed, and their relationships are studied. In one generalization, which was motivated by economics, the main assumption is that $(B - A)^{-1}A$ is nonnegative. In the second generalization, the main assumption is that there exists a matrix $X \geqslant 0$ such that $A = BX$. The equivalence of these two assumptions when B is nonsingular is considered. For $\rho(|B^{-1}A|) < 1$, a complete characterization, involving a condition on the digraph of $B^{-1}A$, is proved. It is conjectured that the characterization holds for $\rho(B^{-1}A) < 1$, and partial results are given for this case.

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1. Introduction

In a recent paper [1] a new generalization of the theorem of Perron and Frobenius to matrix pencils was introduced. For a generalized eigenvalue problem

$$Ax = \lambda Bx \tag{1.1}$$

with $B - A$ nonsingular and $(B - A)^{-1}A$ (entrywise) nonnegative and irreducible, it was shown that there exists $\lambda \in (0, 1)$ and a positive vector x such that $Ax = \lambda Bx$. An analysis of the reducible case was also given. The eigenvalue λ associated with the nonnegative eigenvector is the maximum real eigenvalue in (0, 1). For $(B - A)^{-1}A \geqslant 0$, this eigenvalue is

$$\rho(A, B) := \frac{\rho((B - A)^{-1}A)}{1 + \rho((B - A)^{-1}A)}, \tag{1.2}$$

where for a matrix Z, $\rho(Z) := \max\{|\lambda| \mid Zx = \lambda x\}$ is the classical spectral radius of Z. This result generalizes Perron–Frobenius results for matrix pencils under assumptions motivated by economic models in [7] and [15], but it differs substantially from another generalization of the Perron–Frobenius theory to matrix pencils developed in [11].

A result of [11], Theorem 4.1 states that if

$$\langle B^{\mathrm{T}}y \geqslant 0 \;\text{ implies } A^{\mathrm{T}}y \geqslant 0\rangle \tag{1.3}$$

then there exists a nonnegative eigenvector for Eq. (1.1) corresponding to a nonnegative eigenvalue λ. If furthermore either A or B has full column rank, then this nonnegative λ is equal to the *spectral radius of A relative to B* defined as

$$\rho(A_B) := \begin{cases} \sup\{|\lambda| \mid Ax = \lambda Bx\} & \text{if an eigenvalue of } Ax = \lambda Bx \text{ exists,} \\ -\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

Neither of these two extensions of the Perron–Frobenius theorem is a generalization of the other, as demonstrated by the following examples.

Example 1. If

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then condition (1.3) is satisfied and $\rho(A_B) = 2.4142$. However, the analysis in [1] is not applicable since

$$(B - A)^{-1}A = \begin{bmatrix} -1 & -1 \\ -1/2 & -1 \end{bmatrix}$$

is not nonnegative; in particular, $\rho(A, B)$ is not defined.

Example 2. Let

$$A = \begin{bmatrix} -1/4 & 1/2 \\ 1/2 & 1/8 \end{bmatrix} \quad \text{and} \quad B = I,$$

the identity matrix. Then

$$(B - A)^{-1}A = \begin{bmatrix} 0.0370 & 0.5926 \\ 0.5926 & 0.4815 \end{bmatrix}$$

is nonnegative and $\rho(A, B) = 0.4715$. However, the analysis in [11] is not applicable, since condition (1.3) does not hold when $y \geqslant 0$ and $2y_2 < y_1$. But $\rho(A_B)$ is defined and equals 0.5965.

These examples demonstrate that the values $\rho(A_B)$ and $\rho(A, B)$ may differ, and it may also happen (see [1], Example 3.7) that there exist eigenvalues of Eq. (1.1) of larger modulus than $\rho(A, B)$, while this clearly cannot happen for $\rho(A_B)$. Another major difference is that the results of [11] also extend to rectangular pencils, while [1] makes sense only for square pencils.

It is therefore natural to study the exact relationship between the two generalizations. In [11], it is shown that condition (1.3) is equivalent to the existence of $X \geqslant 0$ such that $A = BX$. Thus, if one considers square pencils and assumes that B^{-1} exists, then the main assumption of [11] is that of the classical Perron-Frobenius theorem, i.e., $Z := B^{-1}A \geqslant 0$, while the main assumption of [1] is that $(B - A)^{-1}A = (I - B^{-1}A)^{-1}B^{-1}A = (I - Z)^{-1}Z \geqslant 0$. So in the simplest possible case, the relationship between the two generalizations should become apparent when the equivalence

$$(I - Z)^{-1}Z \geqslant 0 \iff Z \geqslant 0 \tag{1.5}$$

holds. In particular, when $Z \geqslant 0$ and (1.5) holds, then $\rho(B^{-1}A) = \rho(A_B) = \rho(A, B)$; see Corollary 17. Note that $\rho(A, B)$ is always less than one, so $\rho(Z) < 1$ is a necessary assumption for the equality of $\rho(A, B)$ and $\rho(A_B)$.

Thus, it is an important step in the analysis of the relationship of the two Perron–Frobenius generalizations to study under which conditions the equivalence (1.5) holds. One direction of this equivalence is immediate if $\rho(Z) < 1$.

Proposition 3. *If $\rho(Z) < 1$, then $Z \geqslant 0$ implies that $(I - Z)^{-1}Z \geqslant 0$.*

Proof. As Z is a nonnegative matrix with $\rho(Z) < 1$, the matrix $I - Z$ is an $\mathcal{M}$-matrix. Thus $(I - Z)^{-1} \geqslant 0$ (see, e.g., [2], p. 137) and hence the product $(I - Z)^{-1}Z \geqslant 0$. $\square$

Observe that the other direction in Proposition 3 is not true in general, as shown in Example 2.

The main topic of this paper is the study of the reverse direction in Proposition 3. In Section 3, we give a complete characterization under the assumption $\rho(|Z|) < 1$. Here $|Z|$ denotes the entrywise absolute value of Z. We conjecture that the same characterization also holds in the case $\rho(Z) < 1$, but we have a proof only in some special cases, which are discussed in Section 4. Concluding comments are given in Section 5.

2. Notation and preliminaries

To study the backwards implication in (1.5) we need some concepts from graph theory.

If $Z \in \mathbb{R}^{n,n}$, then entries of Z are denoted by z_{ij}, and we denote by $Z(i_1, i_2, \ldots, i_r)$ the submatrix of Z obtained by deleting rows and columns $i_1, i_2, \ldots, i_r$. If Z is a block partitioned matrix, then Z_{ij} denotes a block submatrix of Z. However, for example, $(I - Z)_{ij}$ is used to denote either an entry or a block submatrix of $I - Z$. Let $\mathscr{D}(Z)$ be the weighted digraph associated with Z, i.e., $\mathscr{D}(Z)$ has vertex set $\{1, 2, \ldots, n\}$ and an arc from i to j weighted as z_{ij} iff $z_{ij} \neq 0$. Directed walks, paths and cycles in $\mathscr{D}(Z)$ are defined in the usual way, see, e.g. [3,13]. A *walk product* (*path product, cycle product*, resp.) is the product of the z_{ij} corresponding to the arcs of the walk (path, cycle, resp.). In particular $\mathscr{D}(Z)$ has a 1-cycle at vertex i with cycle product z_{ii} iff $z_{ii} \neq 0$.

The *corresponding undirected graph* of $\mathscr{D}(Z)$ has vertex set $\{1, 2, \ldots, n\}$ and an arc between i and j iff $z_{ij} \neq 0$ or $z_{ji} \neq 0$. If there exists a path between every pair of distinct vertices, then this graph is connected; otherwise it has at least two *connected components*.

Definition 4. We say that $\mathscr{D}(Z)$ is *arc unique* if, for all vertices i, j with an arc from i to j, this arc is the unique directed path from vertex i to vertex j in $\mathscr{D}(Z)$.

Note that this definition allows $i = j$. We remark that a unipathic digraph with no 1-cycles is arc unique; see, e.g. [12,14]. In general, an arc unique digraph is neither unipathic nor acyclic.

For any $Z \in \mathbb{R}^{n,n}$, there exists a permutation matrix P so that PZP^{T} is in Frobenius normal form, i.e.,

$$PZP^{\mathrm{T}} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1k} \\ & Z_{22} & \cdots & Z_{2k} \\ & & \ddots & \vdots \\ 0 & & & Z_{kk} \end{bmatrix}, \tag{2.1}$$

where Z_{ss} is square and irreducible for $1 \leqslant s \leqslant k$. Note that any 1×1 matrix is irreducible. If $k = 1$, then $\mathscr{D}(Z)$ is strongly connected and in this case (when $n \geqslant 2$) $\mathscr{D}(Z)$ is arc unique iff $\mathscr{D}(Z)$ is minimally strong, see [3], p. 61. The diagonal blocks Z_{ss} correspond to the strongly connected components of $\mathscr{D}(Z)$; see, e.g. [3], p. 58. Note that if $\mathscr{D}(Z)$ is arc unique and if there is a 1-cycle at vertex i, then the strongly connected component of $\mathscr{D}(Z)$ containing vertex i is of order 1.

We are interested in the nonnegativity of $(I - Z)^{-1}Z$, where $I - Z$ is assumed to be nonsingular. Since $(I - Z)^{-1}Z \geqslant 0$ precisely when $(I - PZP^{\mathrm{T}})^{-1}PZP^{\mathrm{T}} \geqslant 0$, we assume w.l.o.g. that Z is in Frobenius normal form (2.1) and we have the following lemma.

Lemma 5. *Assume that Z is in Frobenius normal form* (2.1) *and that $I - Z$ is nonsingular. Then $(I - Z)^{-1}Z$ has $(I - Z_{ss})^{-1}Z_{ss}$ as the s-th diagonal block for $1 \leqslant s \leqslant k$, and for any off diagonal entry $((I - Z)^{-1}Z)_{ij} = (I - Z)^{-1}_{ij}$ for all $i \neq j$.*

Proof. The first statement can be verified by block multiplication. For the second statement, let $Q = I - Z$. Then $(I - Z)^{-1}Z = Q^{-1} - I$, whereas $(I - Z)^{-1} = Q^{-1}$. Thus the two matrices agree off the main diagonal. □

3. The case $\rho(|Z|) < 1$

In this section we describe necessary and sufficient conditions for $Z \geqslant 0$ to be equivalent to $(I - Z)^{-1}Z \geqslant 0$. We study this equivalence in the case that $\rho(|Z|) < 1$. Since the logical structure of the result is quite complicated, we break it into separate theorems.

Theorem 6. *Let $Z \in \mathbb{R}^{n,n}$ with $\rho(|Z|) < 1$ and $\mathscr{D}(Z)$ arc unique. Then $(I - Z)^{-1}Z \geqslant 0$ implies that $Z \geqslant 0$.*

Proof. Since $\rho(|Z|) < 1$ and $\rho(Z) \leqslant \rho(|Z|)$ [8], Theorem 8.1.18, [9], p. 49, it follows that $\rho(Z) < 1$. So $I - Z$ is positive stable, i.e., has all eigenvalues in the right half plane. Hence $I - Z$ is nonsingular and $\det(I - Z) > 0$. The matrix $|Z|$ is nonnegative, and thus by the Perron–Frobenius theorem (see, e.g. [8]), $\rho(|Z(i,j)|) \leqslant \rho(|Z|)$. Hence

$$\rho(Z(i,j)) \leqslant \rho(|Z(i,j)|) < 1$$

and

$$\det(I - Z(i,j)) = \det((I - Z)(i,j)) > 0.$$

Consider an entry $z_{ij} \neq 0$ with $i \neq j$. Then by [13], Corollary 9.1 the matrix entry

$$(I - Z)^{-1}_{ij} = (-1)(-z_{ij})\frac{\det((I - Z)(i,j))}{\det(I - Z)},$$

since arc uniqueness means that the arc from i to j is the unique path from vertex i to vertex j. By Lemma 5 and the positivity of both determinants, $((I-Z)^{-1}Z)_{ij} = \alpha z_{ij}$ where $\alpha > 0$. By assumption $((I-Z)^{-1}Z)_{ij} \geqslant 0$, and thus $z_{ij} > 0$ (since $z_{ij} \neq 0$). If $z_{ii} \neq 0$, then arc uniqueness and Lemma 5 imply that $z_{ii}/(1-z_{ii}) \geqslant 0$, and $\rho(Z) < 1$ implies that $1 - z_{ii} > 0$, giving $z_{ii} > 0$. Thus all nonzero entries of Z are positive. □

The following is the converse of Proposition 3 and Theorem 6. The symbol $\cong$ denotes digraph isomorphism.

Theorem 7. *For a fixed digraph* $\mathbf{D}$, *let* $\mathscr{Z}_{\mathbf{D}} = \{Z \in \mathbb{R}^{n,n} \mid \mathscr{D}(Z) \cong \mathbf{D}$ *and* $\rho(|Z|) < 1\}$. *If the equivalence* $\langle (I-Z)^{-1}Z \geqslant 0 \Longleftrightarrow Z \geqslant 0\rangle$ *holds for all* $Z \in \mathscr{Z}_{\mathbf{D}}$, *then* $\mathbf{D}$ *is arc unique.*

Proof. We prove the contrapositive: *There exists* $Z \ngeqslant 0$ *with* $\rho(|Z|) < 1$ *and* $\mathscr{D}(Z)$ *not arc unique having* $(I-Z)^{-1}Z \geqslant 0$. Considering the irreducible case first, let $\hat{Z}$ be irreducible. Then $\mathscr{D}(\hat{Z})$ is strongly connected, but is assumed not arc unique. Let $\mathscr{D}(\tilde{Z})$ be a strongly connected subgraph of $\mathscr{D}(\hat{Z})$ on n vertices that is arc unique and let $\tilde{Z}$ be an appropriately scaled adjacency matrix so that $\rho(\tilde{Z}) < 1$. Let $\bar{Z}$ be the matrix that has a 1 in each entry where the adjacency matrices of $\mathscr{D}(\tilde{Z})$ and $\mathscr{D}(\hat{Z})$ differ and zeros elsewhere. For sufficiently small $\epsilon > 0$, set $Z = \tilde{Z} - \epsilon\bar{Z}$ so that $\rho(|Z|) < 1$. Notice that $\tilde{Z} \geqslant 0$ but $Z \ngeqslant 0$. Since $\mathscr{D}(Z) \cong \mathscr{D}(\hat{Z})$, $\mathscr{D}(Z)$ is not arc unique. Now, as $\rho(|Z|) < 1$ implies $\rho(Z) < 1$, the Neumann expansion gives

$$\begin{aligned}(I-Z)^{-1}Z &= Z + Z^2 + Z^3 + \cdots \\ &= (\tilde{Z} - \epsilon\bar{Z}) + (\tilde{Z} - \epsilon\bar{Z})^2 + (\tilde{Z} - \epsilon\bar{Z})^3 + \cdots \\ &= (\tilde{Z} + \tilde{Z}^2 + \tilde{Z}^3 + \cdots) + \mathrm{O}(\epsilon) \\ &= (I-\tilde{Z})^{-1}\tilde{Z} + \mathrm{O}(\epsilon),\end{aligned}$$

since $\rho(\tilde{Z}) < 1$. But $(\tilde{Z}^g)_{ij}$ is equal to the sum of all walk products from i to j of length g in $\mathscr{D}(\tilde{Z})$. Consider vertices i and j (not necessarily distinct). As $\tilde{Z}$ is irreducible, for each pair i, j, there exists a path from i to j of some length, say h. Since $\tilde{Z} \geqslant 0$, each path product in $\mathscr{D}(\tilde{Z})$ is positive so $(\tilde{Z}^h)_{ij} > 0$. Because only zero and positive terms occur in the sum of walk products, this implies $((I-\tilde{Z})^{-1}\tilde{Z})_{ij} > 0$. Thus $(I-\tilde{Z})^{-1}\tilde{Z}$ is a positive matrix. Therefore, for ϵ sufficiently small, $(I-Z)^{-1}Z > 0$.

Now consider the reducible case. Let $\hat{Z}$ be reducible and w.l.o.g. assume that $\mathscr{D}(\hat{Z})$ is in Frobenius normal form as in Eq. (2.1). Also assume that $\mathscr{D}(\hat{Z})$ is not arc unique. Let $\tilde{Z}$ be a submatrix of $\hat{Z}$ constructed as follows. For $1 \leqslant s \leqslant k$, diagonal blocks $\tilde{Z}_{ss}$ are constructed as in the irreducible case. Initialize $\tilde{Z}_{pq} = |\hat{Z}_{pq}|$ for all $p \neq q$. For each block $\tilde{Z}_{pq}$, $p < q$, that contains more than one nonzero entry, redefine $\tilde{Z}_{pq}$ by deleting all but one nonzero entry. If there now exists an arc and a path between blocks p and q in $\mathscr{D}(\tilde{Z})$, then redefine $\tilde{Z}_{pq}$ to be the zero matrix. Then $\mathscr{D}(\tilde{Z})$ is arc unique. For $\epsilon > 0$, set $Z = \tilde{Z} - \epsilon\bar{Z}$ where $\bar{Z}$ is the matrix that has a 1 in each entry where the adjacency matrices of $\mathscr{D}(\hat{Z})$ and $\mathscr{D}(\tilde{Z})$ differ and zeros elsewhere. Now $\tilde{Z} \geqslant 0$ but $Z \ngeqslant 0$. Since $I - Z$ and Z are block upper triangular, clearly the p,q block of $(I - Z)^{-1}Z$ vanishes if $p > q$; while for $p < q$ it can be verified by block multiplication that

$$((I - Z)^{-1}Z)_{pq} = \sum_{p=r_1<\cdots<r_m=q} (I - Z_{r_1,r_1})^{-1} Z_{r_1,r_2} (I - Z_{r_2,r_2})^{-1} \times Z_{r_2,r_3} \cdots Z_{r_{m-1},r_m} (I - Z_{r_m,r_m})^{-1}.$$

For the diagonal blocks, either $(I - Z_{pp})^{-1} Z_{pp} = (I - \tilde{Z}_{pp})^{-1}\tilde{Z}_{pp}$, or as in the irreducible case, $(I - Z_{pp})^{-1} Z_{pp} = (I - \tilde{Z}_{pp})^{-1}\tilde{Z}_{pp} + \mathrm{O}(\epsilon)$, depending on whether or not $\mathscr{D}(Z_{\mathrm{pp}})$ is arc unique. Thus, for $p < q$,

$$((I - Z)^{-1}Z)_{pq} = \sum_{p=r_1<r_2<\cdots<r_m=q} (I - \tilde{Z}_{r_1,r_1})^{-1} \tilde{Z}_{r_1,r_2} (I - \tilde{Z}_{r_2,r_2})^{-1} \times \tilde{Z}_{r_2,r_3} \cdots \tilde{Z}_{r_{m-1},r_m} (I - \tilde{Z}_{r_m,r_m})^{-1} + \mathrm{O}(\epsilon). \tag{3.1}$$

For fixed $p < q$, either $((I - Z)^{-1}Z)_{pq} = 0$ or, as $\mathscr{D}(\tilde{Z})$ is arc unique, exactly one summand in the summation in Eq. (3.1) is nonzero. In the latter case, this summand is a positive matrix because $\tilde{Z}_{r_u,r_{u+1}} \geqslant 0$ if $r_u < r_{u+1}$ and $(I - \tilde{Z}_{ss})^{-1} > 0$ for all $1 \leqslant s \leqslant k$ (since $\tilde{Z}_{ss}$ in nonnegative, irreducible and has $\rho(\tilde{Z}_{ss}) < 1$). Using the continuity of the spectral radius (see, e.g. [5,6]), for all ϵ sufficiently small $((I - Z)^{-1}Z)_{pq} > 0$ and $\rho(|Z|) < 1$. □

The construction in Theorem 7 is illustrated in the following example.

Example 8. Let $\hat{Z}$ be a fixed matrix with $\mathscr{D}(\hat{Z})$ having four strongly connected components as shown in Fig. 1. Clearly $\mathscr{D}(\hat{Z})$ is not arc unique. Following the proof given in Theorem 7,

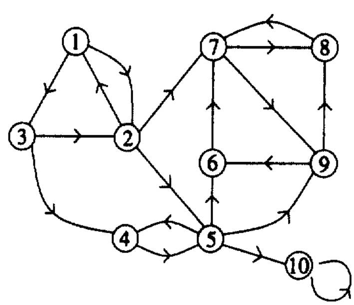

Fig. 1. $\mathscr{D}(\hat{Z})$ of Example 8.

$$Z = \tilde{Z} - \epsilon \bar{Z} = \left[\begin{array}{ccc|cc|cccc|c} 0 & -\epsilon & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & -\epsilon & 0 & -\epsilon & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & -\epsilon & 0 & 0 & 6 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0.7 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{array}\right]$$

where $\epsilon > 0$ and the entries in each diagonal block $\tilde{Z}_{ss}$, $1 \leqslant s \leqslant 4$, have been scaled so that $\rho(\tilde{Z}_{ss}) < 1$. The $-\epsilon$ in the $(1,2)$ position occurs because removal of the arc from vertex 1 to vertex 2 makes $\mathscr{D}(\tilde{Z}_{11})$ arc unique. The $-\epsilon$ in the $(2,5)$ position occurs so that the redefined $\tilde{Z}_{12}$ has only one nonzero entry. The $-\epsilon$ in the $(2,7)$ position occurs because $\tilde{Z}_{13}$ must be redefined to be 0, since there is a path between blocks 1 and 3. Setting $\epsilon = 0.1$ gives $\rho(|Z|) = 0.9333 < 1$ and $(I - Z)^{-1}Z \geqslant 0$. Note that $Z \ngeqslant 0$.

We end this section by collecting together the results of Proposition 3, and Theorems 6 and 7.

Theorem 9. *For a fixed digraph* $\mathbf{D}$, *let* $\mathscr{Z}_{\mathbf{D}} = \{Z \in \mathbb{R}^{n,n} \mid \mathscr{D}(Z) \cong \mathbf{D}$ *and* $\rho(|Z|) < 1\}$. *Then the following are equivalent:*

(i) $(I - Z)^{-1}Z \geqslant 0 \iff Z \geqslant 0$ *for all* $Z \in \mathscr{Z}_{\mathbf{D}}$.

(ii) $\mathbf{D}$ *is arc unique.*

To illustrate the logic of Theorem 9, consider the following example.

Example 10. Let

$$Z_1 = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/4 \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/4 \end{bmatrix}.$$

Then $\rho(|Z_1|) = \rho(|Z_2|) = 0.8431$. Also,

$$(I - Z_1)^{-1}Z_1 = \begin{bmatrix} 2/3 & 2/3 \\ 4/3 & 1/3 \end{bmatrix} \quad \text{and} \quad (I - Z_2)^{-1}Z_2 = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}.$$

Here $\mathscr{D}(Z_1) \cong \mathscr{D}(Z_2)$ is not arc unique, and note that the equivalence in (i) of Theorem 9 holds for Z_2 but not for Z_1.

4. The case $\rho(Z) < 1$

In this section, we consider the equivalence of (i) and (ii) in Theorem 9 when the spectral radius condition is relaxed to $\rho(Z) < 1$. For this case we have the following partial result.

Proposition 11. *Let $\rho(Z) < 1$ and $\mathscr{D}(Z)$ be arc unique with all cycle products positive. Then $(I - Z)^{-1}Z \geqslant 0$ iff $Z \geqslant 0$.*

Proof. If $\mathscr{D}(Z)$ has all cycle products positive, then each irreducible block $Z_{ss}, 1 \leqslant s \leqslant k$, is signature similar to $|Z_{ss}|$; see, e.g. [4]. Since $\rho(Z) = \max_{s=1,2,\ldots,k} \rho(Z_{ss})$, it follows that $\rho(|Z|) = \rho(Z)$ (see also [9], p. 50) and the result follows from Theorem 6 and Proposition 3. $\square$

We conjecture that the positive cycle condition in Proposition 11 is not required; thus we have the following (cf. Theorem 9).

Conjecture 12. For a fixed digraph $\mathbf{D}$, let $\mathscr{Z}'_{\mathbf{D}} = \{Z \in \mathbb{R}^{n,n} \mid \mathscr{D}(Z) \cong \mathbf{D}$ and $\rho(Z) < 1\}$. Then the following are equivalent:

(i) $(I - Z)^{-1}Z \geqslant 0 \iff Z \geqslant 0$ for all $Z \in \mathscr{Z}'_{\mathbf{D}}$.

(ii) $\mathbf{D}$ is arc unique.

Note that if under our stated conditions on $Z, (I - Z)^{-1}Z$, and $\mathscr{D}(Z)$, we have that $\rho(Z) < 1$ implies that $\rho(|Z|) < 1$, then Conjecture 12 follows from Theorem 9; we conjecture that this implication holds.

We now give two additional results in which a digraph condition is given that is sufficient (but not necessary) for the equivalence (i) of Conjecture 12 to hold.

Proposition 13. *If $\rho(Z) < 1, \mathscr{D}(Z)$ is arc unique, and each arc in $\mathscr{D}(Z)$ has at least one incident vertex with outdegree or indegree exactly 1, then $(I - Z)^{-1}Z \geqslant 0$ implies that $Z \geqslant 0$.*

Proof. Consider $z_{ij} \neq 0$ where z_{ij} is an entry in an off-diagonal block Z_{pq} of Eq. (2.1). Then

$$(I - Z)^{-1}_{ij} = \frac{z_{ij} \det((I - Z)(i,j))}{\det(I - Z)},$$

by [13], Corollary 9.1, since $\mathscr{D}(Z)$ is arc unique. As i and j are in different irreducible blocks, Z_{pp} and Z_{qq} respectively,

$$(I-Z)^{-1}_{ij} = \frac{z_{ij}\ \det((I-Z_{pp})(i))\det((I-Z_{qq})(j))}{\det(I-Z)} \prod_{\substack{s=1,2,\dots,k\\ s\neq p,q}} \det(I-Z_{ss})$$
$$= \frac{z_{ij}\ \det((I-Z_{pp})(i))}{\det(I-Z_{pp})}\frac{\det((I-Z_{qq})(j))}{\det(I-Z_{qq})}$$
$$= z_{ij}(I-Z_{pp})^{-1}_{ii}(I-Z_{qq})^{-1}_{jj},$$

by using the adjoint formula for each inverse. But $(I-Z_{ss})^{-1} \geqslant I$, from the proof of Lemma 5, thus $(I-Z)^{-1}_{ij} = ((I-Z)^{-1}Z)_{ij} \geqslant 0$ implies that $z_{ij} > 0$.

If $z_{ii} \neq 0$, then as in the proof of Theorem 6, $z_{ii} > 0$. It suffices now to consider an irreducible block of order $\geqslant 2$. Suppose that vertex j has indegree 1, and $z_{ij} \neq 0$, where z_{ij} belongs to an irreducible block Z_{ss} and $i \neq j$. Then

$$((I-Z)^{-1}Z)_{ij} = \sum_l (I-Z)^{-1}_{il} z_{lj}$$

where the summation is over the rows (and columns) in Z_{ss}. By the assumed conditions on $\mathscr{D}(Z)$, this sum has only one term, namely $(I-Z)^{-1}_{ii}z_{ij}$. But $(I-Z)^{-1}_{ii} \geqslant 1$, thus $(I-Z)^{-1}Z \geqslant 0$ implies that $z_{ij} > 0$. Similarly, if vertex i has outdegree 1 and $z_{ij} \neq 0$, then

$$(Z(I-Z)^{-1})_{ij} = z_{ij}(I-Z)^{-1}_{jj}.$$

Since $Z(I-Z)^{-1} = (I-Z)^{-1}Z \geqslant 0$, this implies that $z_{ij} > 0$. Hence $Z \geqslant 0$. $\square$

Vertex i is a *cut vertex* if the corresponding undirected graph of $\mathscr{D}(Z) - \{i\}$ has more connected components than the corresponding undirected graph of $\mathscr{D}(Z)$. We define a *leaf cycle* in $\mathscr{D}(Z)$ as a cycle of length $\geqslant 2$ with exactly one cut vertex. If $\mathscr{D}(Z)$ is arc unique, then each arc on a leaf cycle satisfies the extra digraph condition of Proposition 13. By the method in the proof of Proposition 13, if $\rho(Z) < 1$ and $(I-Z)^{-1}Z \geqslant 0$, then $z_{ij} > 0$ when arc i,j lies on a leaf cycle of $\mathscr{D}(Z)$. Thus, under the stated conditions, each leaf cycle has positive cycle product.

Proposition 14. *If $\rho(Z) < 1$, $\mathscr{D}(Z)$ is arc unique and has no cycle of length greater than 2, then $(I-Z)^{-1}Z \geqslant 0$ implies that $Z \geqslant 0$.*

Proof. Firstly, consider the case when Z is irreducible, thus $I-Z$ is irreducible. The assumption $(I-Z)^{-1}Z \geqslant 0$ and Lemma 5 imply that $(I-Z)^{-1} \geqslant 0$. Also, the assumption $\rho(Z) < 1$ implies that $I-Z$ is positive stable. Since $\mathscr{D}(Z)$ is assumed to have 2 as the length of it longest cycle, $I-Z$ is an $\mathscr{M}$-matrix by [10], Theorem 2. Hence $Z \geqslant 0$.

When Z is reducible, take it in Frobenius normal form (2.1) with $k \geqslant 2$. Lemma 5 and the above proof show that all entries in the diagonal blocks

Z_{ss} are nonnegative. Consider $z_{ij} \neq 0$ where z_{ij} is an entry in an off-diagonal block Z_{pq}. The first part of the proof of Proposition 13 shows that $z_{ij} > 0$. □

We conclude this section by using positivity of leaf cycles in two examples for which the conjecture is true.

Example 15. Let Z be a fixed matrix with $\rho(Z) < 1$ and $\mathscr{D}(Z)$ a cycle of length $t \geqslant 2$ with leaf cycles of any length attached to at most $t-1$ vertices on the cycle. Note that $\mathscr{D}(Z)$ is arc unique. Assuming that $(I-Z)^{-1}Z \geqslant 0$, the entries of Z corresponding to leaf cycle edges can be proved positive by the method in Proposition 13 (see the discussion above Proposition 14). For the cycle of length t, assume w.l.o.g. that vertices $1, 2, \ldots, t$ lie on the cycle in that order, with vertex 1 having indegree 1. Consider the matrix entry

$$((I-Z)^{-1}Z)_{11} = \frac{(I-Z)^{-1}_{1t} z_{t1}}{\det(I-Z)}$$
$$= \frac{(-1)^{t-1}(-z_{12})(-z_{23})\cdots(-z_{t-1,t})z_{t1}\ \det((I-Z)(1,2,\ldots,t))}{(\det(I-Z))^2}$$

by [13], Corollary 9.1. Here $\det((I-Z)(1,2,\ldots,t)) = 1$, since removing vertices $1, 2, \ldots, t$ breaks every cycle in $\mathscr{D}(Z)$. Thus $((I-Z)^{-1}Z)_{11} \geqslant 0$ implies that the t-cycle product is positive. Thus by Proposition 11, $Z \geqslant 0$.

Example 16. Let $Z \in \mathbb{R}^{n,n}$ be a fixed real matrix with $\rho(Z) < 1$ and $\mathscr{D}(Z)$ as in Fig. 2, and note that $\mathscr{D}(Z)$ is arc unique. Assuming that $(I-Z)^{-1}Z \geqslant 0$, as in Example 15 the entries of Z corresponding to the 3 leaf cycles can be shown positive. From the proof of Lemma 5, $(I-Z)^{-1}Z \geqslant 0$ implies that $(I-Z)^{-1} \geqslant I$. The inequalities on the entries $(I-Z)^{-1}_{33} \geqslant 1$ and $(I-Z)^{-1}_{44} \geqslant 1$

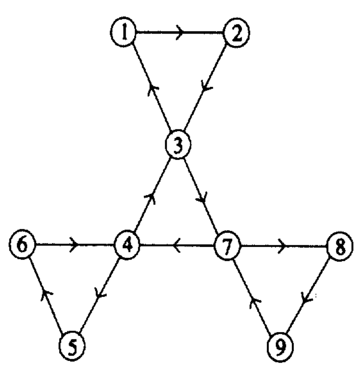

Fig. 2. $\mathscr{D}(Z)$ of Example 16.

imply that $(1 - z_{12}z_{23}z_{31})$, $(1 - z_{45}z_{56}z_{64})$ and $(1 - z_{78}z_{89}z_{97})$ are either all less than 1 or all greater than 1. Since $|\det Z| = |z_{12}z_{23}z_{31}z_{45}z_{56}z_{64}z_{78}z_{89}z_{97}| < 1$, each of the above three terms must lie in $(0, 1)$. Now the inequalities $((I - Z)^{-1}Z)_{37} \geqslant 0$, $((I - Z)^{-1}Z)_{74} \geqslant 0$ and $((I - Z)^{-1}Z)_{43} \geqslant 0$, respectively, give $z_{37} \geqslant 0$, $z_{74} \geqslant 0$, and $z_{43} \geqslant 0$. Thus $Z \geqslant 0$.

5. Concluding comments

Direct consequences of the previous results are the following corollaries that give sufficient conditions so that both generalizations of the Perron–Frobenius theorem lead to the same spectral radius.

Corollary 17. *Let $A, B \in \mathbb{R}^{n,n}$ and suppose that B and $B - A$ are nonsingular. Assume further that $Z = B^{-1}A$ is nonnegative and $\rho(Z) < 1$. Then $\rho(A, B) = \rho(A_B)$.*

Proof. By Proposition 3, under the given assumptions $Z \geqslant 0$ implies that $(I - Z)^{-1}Z \geqslant 0$, and hence $(B - A)^{-1}A \geqslant 0$. Thus $\rho(A, B)$ as in Eq. (1.2) is defined and

$$\rho(A, B) = \frac{\mu}{1 + \mu},$$

where $\mu = \rho((B - A)^{-1}A)$. Also, there exists a nonnegative vector x such that

$$(B - A)^{-1}Ax = \mu x$$

which (see [1]) is equivalent to

$$Ax = \frac{\mu}{1 + \mu}Bx$$

from which it follows that $\rho(A_B) = (\mu/(1 + \mu))$. $\square$

Corollary 18. *Let $A, B \in \mathbb{R}^{n,n}$ and suppose that B and $B - A$ are nonsingular. Assume further that $(I - Z)^{-1}Z$ is nonnegative, where $Z = B^{-1}A$, that $\rho(|Z|) < 1$ and that $\mathscr{D}(Z)$ is arc unique. Then $\rho(A, B) = \rho(A_B)$.*

Proof. By Theorem 6, under the given assumptions, $Z \geqslant 0$. Thus by the proof of Corollary 17, $\rho(A, B) = \rho(A_B)$. $\square$

Note that if Conjecture 12 is true, the assumption $\rho(|Z|) < 1$ in Corollary 18 can be replaced by $\rho(Z) < 1$.

Finally, it should be noted that the results in [1] can be easily generalized by introducing scaling parameters. If positive α, β exist such that $\beta B - \alpha A$ is nonsingular and $(\beta B - \alpha A)^{-1}A$ is nonnegative, then analogous results are obtained by replacing $\rho(Z) < 1$ by $\rho(Z) < (\beta/\alpha)$ and $\rho(A,B)$ by

$$\rho_{\alpha,\beta}(A,B) := \frac{\beta\rho(((\beta B - \alpha A)^{-1})A)}{1 + \alpha\rho(((\beta B - \alpha A)^{-1})A)}.$$

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